

# Black hole phase transitions via Bragg-Williams

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## ABSTRACT

We argue that a convenient way to analyze instabilities of black holes in AdS space is via Bragg-Williams construction of a free energy function. Starting with a pedagogical review of this construction in condensed matter systems and also its implementation to Hawking-Page transition, we study instabilities associated with hairy black holes and also with the  $R$ -charged black holes. For the hairy black holes, an analysis of thermal quench is presented.

December 2010

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# 1 Introduction

Within the mean field approximation, phase transition is primarily described via Landau theory. Under the assumptions that the order parameter is small and uniform near the transition, this theory provides us with a wealth of information about the nature of the phase transition. It is based upon a power series expansion of free energy in terms of the order parameter. The terms in this expansion are normally determined by symmetry considerations of the phases. Furthermore, owing to the smallness of the order parameter, only a few leading terms are kept. The usefulness of the Landau theory lies in its simplicity as most of its predictions can be achieved by solving simple algebraic equations [1]. While this theory is most suitable in describing a second order phase transition, one needs to be somewhat careful to treat first order phase transition within this framework. This is because, in a first order transition, order parameter suffers a discontinuous jump across the critical temperature. If this change is large, a power series expansion of free energy may acquire ambiguities. One then requires a more complete mean field theory. An example of this kind is the Bragg-Williams (BW) theory [2, 3]. Originally used to describe order - disorder transition of alloys, it has a wide range of applications [1, 4]. In this approach, one constructs an approximate expression for the free energy in terms of the order parameter and uses the condition that its equilibrium value minimizes the free energy. Our aim in this paper is to use this approach to study phase transition involving black holes in the presence of a negative cosmological constant.

In 1970, following the work of Bekenstein and others, Hawking showed that stationary black holes have temperature  $T$  related to their surface gravity, and, they behave like thermodynamic objects. In the presence of a negative cosmological constant, these black holes asymptote to the anti-de Sitter (AdS) space and, provided that their size is *sufficiently large*, the temperature typically increases with their internal energy. Therefore, unlike black holes in Minkowski space, these have positive specific heat and hence they are thermodynamically stable. However, as we reduce the temperature, at some stage, various instabilities creep in. A prime example of this kind is the well known Hawking-Page (HP) instability [5], where below a critical temperature, a AdS-Schwarzschild black hole becomes unstable and crosses over to the thermal AdS space via a first order phase transition. A similar behaviour arises for the Reissner-Nordström black holes in AdS space in the grand canonical ensemble [6, 7]. Subsequently, it was found that the charged black holes in five dimensional  $\mathcal{N} = 2$  gauged supergravity theory also exhibit rich phase structures. Black holes in this theory, known as  $R$ -charged black holes, can carry three independent gauge charges and the stability of these black holes were studied, for example, in [8–10]. For single  $R$ -charged holes, the phase structure is shown in figure 1. It is plotted in the  $T - \mu$  plane where  $\mu$  is the chemical potential conjugate to the charge. There are three distinct phases, namely, the thermal AdS, black hole and a yet unknown phase. At a low temperature and small chemical potential, the system is always in thermal

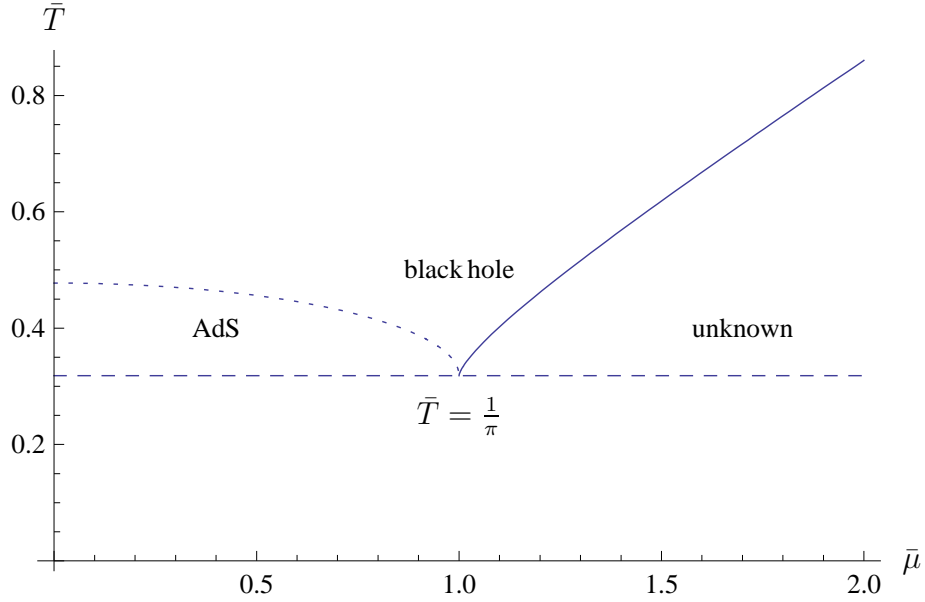


Figure 1: Phase diagram for the  $R$ -charged black hole with single charge shown in temperature, chemical potential plane. Line separating thermal AdS and black hole represents the first order phase transition line given by equation (40). On the other hand, the line between black hole and the unknown phase is a second order line - the equation of which is given in (41). The dashed line is for  $\bar{T} = 1/\pi$  below which we can not extend various phases.

AdS phase. The cross-over from AdS to the black hole phase is shown by the dotted line in the plot. This is the usual first order HP transition. The black hole phase at fixed temperature also becomes unstable once the chemical potential is increased beyond a critical value. The corresponding stable phase is unknown as yet<sup>5</sup>. However, if a stable phase exists, this transition would be a continuous phase transition marked by divergences of specific heat and susceptibility. The solid line in figure 1 represents this critical line. Another interesting example of phase transition in black hole physics includes holes with scalar hair. In AdS space, it is possible to construct black holes with hair. The one which we consider here was found in [11]. These are electrically charged black hole solutions in four dimensional AdS space with a conformally coupled scalar. Unlike previous examples, here, the horizon is a negatively curved two dimensional constant curvature manifold. We will call these holes as hairy black holes in this paper. A recent analysis on the stability of these black holes were carried out in [12]. It was concluded that at low temperature, the hairy black hole is stable. At high temperature, by losing “hair”, it becomes unstable and crosses over to

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<sup>5</sup>It may also be possible that there is no stable phase at all.

a Reissner-Nordström phase with same horizon topology. Our aim in this paper is to study various instabilities mentioned in the previous paragraph within the framework of BW theory. As we will see, all these transitions are very elegantly captured within this scheme. Its implementation is simple, it makes definitive statements about the nature of the phase transition and it gives us the mean field critical exponents where the black hole undergoes a continuous transition. In particular, for hairy to Reissner-Nordström transition, we construct the off-shell free energy function in terms of a suitably chosen order parameter and study the behaviour around its saddle points. We further use the BW potential to analyze the system under temperature quench. It turns out that the time variation of the order parameter after quench from unstable to its stable minimum can be semi-analytically constructed. As for the  $R$ -charged black holes, we reproduce the complete phase diagram in figure 1 and also compute the classical critical exponents near the second order instability.

Our motivation to analyse phases via BW theory grew out of gauge/gravity correspondence. Within this correspondence, gravity theory in AdS space is expected to have a gauge theory dual on the boundary. Details of these gauge theories depend on how one embeds AdS in ten or eleven dimensions, nature of the compactifications etcetera, and, in many cases, are not explicitly known. However, the dual nature of the correspondence suggests that since the gravity is weakly coupled, the gauge theory, if exists, has to have a strong coupling. Due to the lack of a systematic approach to handle strongly coupled theories, direct gauge theoretic computations become difficult. However, the behaviour of the weakly coupled gravity along with gauge/gravity duality often helps us exploring strongly coupled gauge theories. Association of deconfining transition of large  $N$ ,  $\mathcal{N} = 4$  Yang-Mills with the HP transition in five dimensional AdS space is a classic example in this regard [13]. Our hope is to construct candidate effective potentials for gauge theories describing various phases at non-zero temperature via computations of BW potentials from their gravity duals. For a partial success in this direction for  $R$ -charged black hole with flat horizon, see [14]. Our construction of BW potential for hairy black holes might be useful to study some exotic holographic superconductors with higher order phase instabilities.

The paper is organized as follows. In the next section, we introduce BW construction by considering the Ising model. We then employ this construction for AdS-Schwarzschild and Reissner-Nordström black hole in order to study the HP transition. This section is a review of the known results. In section three, we analyze black holes with of five dimensional  $\mathcal{N} = 2$  gauged supergravity theories with single  $R$ -charge. Besides reproducing the phase diagram given in figure 1, we also find out various critical exponents near its second order instability line. Section four is devoted to the study of phase transition involving hairy black holes of [11]. Via BW construction, we critically analyze physics close to the saddle points representing stable phases. This section also includes an analysis of the system under thermal quench. The BW potential can be re-expressed in terms of the value of the scalar on the horizon. Treating this as an order parameter, we construct a *time-dependent* solution representing the

rolling of the order parameter from unstable to the stable point after the quench. In the last section, besides summarizing our results, we speculate how our results might be useful from the perspective of AdS/CFT correspondence.

## 2 Bragg-Williams construction: a brief review

This section is a review of BW theory and pedagogical in nature. It has three subsections. In the first subsection, we discuss Ising model and use BW theory to capture second order paramagnetic to ferromagnetic transition. The later two subsections describe first order HP transitions for Schwarzschild and Reissner-Nordström black holes in AdS space respectively.

### 2.1 Paramagnetic to ferromagnetic transition

Bragg-Williams construction is perhaps best described via Ising model [1]. Let us consider Ising model on a lattice where, on each site, the classical spin variable  $\sigma_l$  takes values  $\pm 1$ . These spins interact via a nearest neighbour coupling  $J > 0$ . The Hamiltonian is given by

$$H = -J \sum_{\langle ll' \rangle} \sigma_l \sigma_{l'}. \quad (1)$$

Here the sum is over the nearest neighbour  $l$  and  $l'$ . The order parameter is  $m = \langle \sigma \rangle$ , the average of the spin. For spatially uniform  $m$ , the entropy can be computed exactly. The total magnetic moment is

$$m = \frac{N_{+1} - N_{-1}}{N}, \quad (2)$$

where  $N_{+1}$  and  $N_{-1}$  are the total number of  $+1$  and  $-1$  spins respectively. The total number of lattice sites is denoted by  $N$ . The entropy is the logarithm of the number of states and is given by

$$S = \ln({}^N C_{N_{+1}}) = \ln({}^N C_{N(1+m)/2}) \quad (3)$$

which, for entropy per unit spin, gives

$$s(m) = \frac{S}{N} = \ln 2 - \frac{1}{2}(1+m)\ln(1+m) - \frac{1}{2}(1-m)\ln(1-m). \quad (4)$$

In BW theory, the energy  $\langle H \rangle$  is approximated via replacing  $\sigma$  by its position independent average  $m$ . Thus

$$E = -J \sum_{\langle ll' \rangle} m^2 = -\frac{1}{2} J N z m^2, \quad (5)$$

where  $z$  is the number of nearest neighbours in the lattice. One then constructs the BW free energy per spin as

$$\begin{aligned} f(T, m) &= \frac{E - TS}{N} \\ &= -\frac{1}{2}Jzm^2 - T \ln 2 + \frac{T}{2}(1+m)\ln(1+m) + \frac{T}{2}(1-m)\ln(1-m). \end{aligned} \quad (6)$$

The BW free energy  $f(T, m)$  can be plotted as a function of  $m$  for various temperatures. It can be checked that, for  $T > Jz$ , it has a single minimum at  $m = 0$ . However, for  $T < Jz$ , two minima occurs for non-zero values of  $m$  leading to paramagnetic to ferromagnetic transition. Critical temperature ( $T_c$ ) for this second order transition can be found by setting first and second derivatives of  $f$  to zero with the result  $T_c = Jz$ . For more details, we suggest the readers to look at [1, 4].

## 2.2 HP transition: AdS-Schwarzschild black hole

We can implement similar construction for AdS black holes. Consider a Schwarzschild black hole in  $(n+2)$  dimensional AdS space. The metric is given by

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega_n^2, \quad (7)$$

with

$$V(r) = \left(1 - \frac{M}{r^{n-1}} + \frac{r^2}{l^2}\right). \quad (8)$$

Here  $M$  is a parameter related to the mass or internal energy of the black hole and  $l$  is the inverse radius of AdS space. We have set  $(n+2)$  dimensional gravitational constant  $G_{n+2}$  to one. The black hole has a single horizon where  $g_{tt}$  vanishes. We will identify the horizon radius as  $r_+$ . The dimensionless temperature, energy and entropy densities are give by

$$\begin{aligned} \bar{T} = lT &= \frac{(n+1)\bar{r}^2 + (n-1)}{4\pi\bar{r}}, \\ \bar{E} = lE &= \frac{n(\bar{r}^{n+1} + \bar{r}^{n-1})}{16\pi}, \\ \bar{S} &= \frac{\bar{r}^n}{4}. \end{aligned} \quad (9)$$

Here  $l\bar{r} = r_+$ . Before constructing the BW free energy, we will have to decide on an order parameter. Noticing the form of the entropy and the energy, it is only natural to consider  $\bar{r}$  as the order parameter. We will see later that this order parameter has right behaviour expected from the instability associated with this black hole. We are now in a position to construct the BW free energy  $\bar{\mathcal{F}}(\bar{r}, \bar{T})$  as

$$\bar{\mathcal{F}}(\bar{r}, \bar{T}) = \bar{E} - \bar{T}\bar{S} = \frac{n(\bar{r}^{n+1} + \bar{r}^{n-1})}{16\pi} - \bar{T}\frac{\bar{r}^n}{4}. \quad (10)$$

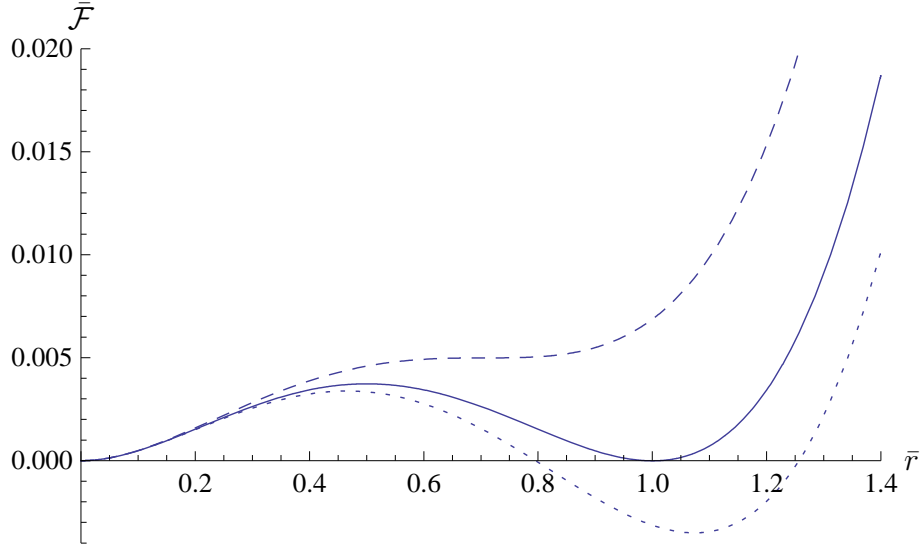


Figure 2: BW free energy for five dimensional AdS-Schwarzschild black holes plotted against horizon radius  $\bar{r}$  for different temperatures  $\bar{T}$ . The solid line has two degenerate minima - representing co-existence of black hole phase (minimum at  $\bar{r} = 1$ ) and the thermal AdS phase (with  $\bar{r} = 0$ ). This happens at a critical temperature  $\bar{T}_c = 3/(2\pi)$ . While above this temperature black hole is stable (dotted line), AdS is a preferred phase below  $\bar{T}_c$  (dashed line).

A plot of the free energy in five dimensions as a function of  $\bar{r}$  for various temperatures is shown in figure 2. Note that in (10), the temperature is a parameter. Its dependence on  $\bar{r}$  as given in (9) appears after minimizing  $\bar{\mathcal{F}}$  with respect to  $\bar{r}$ . At this minimum  $\bar{\mathcal{F}}$  reduces to the on-shell free energy of the black hole. It is given by

$$\bar{F} = \bar{\mathcal{F}}|_{\min} = -\frac{\bar{r}^{n-1}(\bar{r}^2 - 1)}{16\pi}. \quad (11)$$

We identify the AdS free energy with  $\bar{r}$  equals to zero. The first order transition appears when

$$\bar{\mathcal{F}} = 0, \text{ and } \frac{\partial \bar{\mathcal{F}}}{\partial \bar{r}} = 0, \quad (12)$$

are satisfied simultaneously. This happens for

$$\bar{r} = 1, \text{ and } \bar{T}_c = \frac{3}{2\pi}. \quad (13)$$

Below this temperature, black hole phase becomes unstable. As can be seen from the dashed line of figure 2, the  $\bar{r} = 0$  phase is preferred. This is identified as the AdS phase. This is a first order phase transition causing a discontinuous change in the order parameter  $\bar{r}$ .

### 2.3 HP transition: Reissner-Nodström

A similar analysis can be performed for charged black holes in the AdS space. More specifically, we consider here the Reissner-Nodström black holes. Our discussion is a brief review of [15]. The metric has the same form as (7) with  $V(r)$  given by

$$V(r) = \left(1 - \frac{M}{r^{n-1}} + \frac{q^2}{r^{2n-2}} + \frac{r^2}{l^2}\right), \quad (14)$$

where the additional  $q$  dependent term is due to the electric charge that is carried by the configuration. The largest root of the equation  $V(r) = 0$  represents the outer horizon and as before we parametrize it by  $r_+$ . The chemical potential  $\mu$  conjugate to the charge is given by

$$\mu = \frac{q}{cr_+^{n-1}}, \text{ with } c = \sqrt{\frac{2(n-1)}{n}}. \quad (15)$$

The temperature, energy, entropy and charge densities are

$$\begin{aligned} \bar{T} &= \frac{(n-1)(1 - c^2\bar{\mu}^2) + (n+1)\bar{r}^2}{4\pi\bar{r}}, \\ \bar{E} &= \frac{n}{16\pi} \left( \bar{r}^{n-1} + \frac{q^2}{\bar{r}^{n-1}} + \bar{r}^{n+1} \right), \\ \bar{S} &= \frac{\bar{r}^{n-1}}{4}, \\ \bar{Q} &= \sqrt{2n(n-1)}q. \end{aligned} \quad (16)$$

Here quantities with bars are made dimensionless by multiplying appropriate factors of  $l$  wherever necessary. We have, as before, taken  $G_{n+2} = 1$ . The BW free energy density, in the grand canonical ensemble, is then

$$\begin{aligned} \bar{\mathcal{F}}(\bar{r}, \bar{T}, \bar{\mu}) &= \bar{E} - \bar{T}\bar{S} - \bar{Q}\bar{\mu} \\ &= n\bar{r}^{n-1}(1 - c^2\bar{\mu}^2) - 4\pi\bar{r}^n\bar{T} + n\bar{r}^{n+1}. \end{aligned} \quad (17)$$

Qualitative behaviour of the free energy is determined by whether  $\bar{\mu}$  is less than or greater than  $1/c$ . We will only consider the case  $\bar{\mu} < 1/c$ . The other case can be found in [7]. For fixed  $\bar{\mu}$ , one gets a similar graph as in figure 2. At the saddle point of  $\bar{\mathcal{F}}$ , we get

$$\bar{r} = \frac{4\pi\bar{T} + \sqrt{16\pi^2\bar{T}^2 - 4(n-1)(1 - c^2\bar{\mu}^2)}}{2(n+1)}, \quad (18)$$

which, when inverted to get the temperature, reproduces the one in (16). Critical temperature can be found using (12) with the result

$$\bar{T}_c = \frac{n\sqrt{1 - c^2\bar{\mu}^2}}{2\pi}. \quad (19)$$



Having discussed an application of BW formalism in the study of black hole instabilities, in the next section, we discuss another class of black holes which show both first and second order instabilities as  $(\bar{T}, \bar{\mu})$  are tuned and we analyze the system within the above framework.

### 3 $R$ -charged black hole with spherical horizon: Instabilities

Let us start by briefly recapitulating the black hole in five dimensional  $\mathcal{N} = 2$  gauged supergravity. Five dimensional  $\mathcal{N} = 2$  gauged supergravity is obtained by compactification of ten dimensional IIB supergravity on  $S^5$ . As shown in [16], this theory admits asymptotically AdS black hole solutions with three  $U(1)$  charges with three different horizon topology. For the purpose of this note we will focus on singly charged black hole with spherical horizon.

The black hole metric with a single  $U(1)$  charge is given by

$$ds^2 = -H^{-\frac{2}{3}} f dt^2 + H^{\frac{1}{3}} (f^{-1} dr^2 + r^2 d\Omega_3^2), \quad (20)$$

where

$$f = 1 - \frac{m}{r^2} + \frac{r^2}{l^2} H, \quad H = 1 + \frac{q}{r^2}. \quad (21)$$

In the above equation,  $d\Omega_3^2$  is the metric on unit three sphere,  $l$  and  $m$  are related to the cosmological constant and the ADM mass of the black hole. In particular,  $l$  has a dimension of length. The zero of  $f$  gives the location of the horizon and in the above parametrization, the horizon appears at  $r = r_+$  where

$$r_+ = \left( \frac{-l^2 - q + \sqrt{(l^2 + q)^2 + 4ml^2}}{2} \right)^{\frac{1}{2}}. \quad (22)$$

There is a non-trivial gauge field potential associated with this geometry and is given by

$$A_t^i = \frac{\sqrt{q(r_+^2 + q)(1 + r_+^2)}}{r^2 + q}. \quad (23)$$

From the above we see that  $q$  is related to the physical charge. More explicitly, the physical charge

$$Q = \sqrt{q(r_+^2 + q)(1 + r_+^2)}. \quad (24)$$

The chemical potential is defined as the value of  $A_t^i$  at the horizon and is given by

$$\mu = \sqrt{\frac{q(1 + r_+^2)}{r_+^2 + q}}. \quad (25)$$

It will be convenient for us to scale all the dimensionful quantities with appropriate powers of  $l$  and make them dimensionless. We write all these parameters with a bar on the top. For example, the dimensionless horizon radius and Hawking temperature of the black hole are given by,

$$\bar{r} = \frac{r_+}{l}, \quad \bar{q} = \frac{q}{l^2}, \quad \bar{T} = lT = \frac{2\bar{r}^2 + \bar{q} + 1}{2\pi\sqrt{\bar{r}^2 + \bar{q}}}. \quad (26)$$

Furthermore, we define the dimensionless Newton's constant  $\bar{G}$  as  $\bar{G} = l^3 G$  and set  $\bar{G} = \pi/4$ . With this convention, energy and entropy are given by

$$\bar{E} = \frac{3}{2}\bar{m} + \bar{q}, \quad \bar{S} = 2\pi\bar{r}^2\sqrt{\bar{r}^2 + \bar{q}}. \quad (27)$$

We would like to study the system in the grand canonical ensemble where we treat  $\bar{T}$  and  $\bar{\mu}$  as external parameters. The free energy is given by

$$\bar{F} = \bar{E} - \bar{T}\bar{S} - \bar{\mu}\bar{Q} = -\frac{\bar{r}^2(\bar{r}^4 + \bar{\mu}^2 - 1)}{2(\bar{r}^2 - \bar{\mu}^2 + 1)} = -\bar{P}. \quad (28)$$

Here  $\bar{P}$  is the pressure. Let us note that  $\bar{F}$  changes sign when  $\bar{r}^4 + \bar{\mu}^2 - 1$  changes sign. This is a first order transition and it leads to a crossover from AdS phase to the black hole phase. For the gauge theory this represents the deconfining transition. Given all these thermodynamic quantities, it is straightforward to compute the specific heat and susceptibility. These are given respectively by

$$\begin{aligned} \bar{C} &= \left( \bar{T} \frac{\partial \bar{S}}{\partial \bar{T}} \right)_{\bar{\mu}} = \frac{2\pi\bar{r}^2(1 + 2\bar{r}^2 + \bar{q})(3 + 3\bar{r}^2 - \bar{q})\sqrt{\bar{r}^2 + \bar{q}}}{2\bar{r}^4 + \bar{r}^2 + \bar{q}\bar{r}^2 - \bar{q}^2 + 2\bar{q} - 1}, \\ \bar{\chi} &= \left( \frac{\partial \bar{Q}}{\partial \bar{\mu}} \right)_{\bar{T}} = \frac{(\bar{r}^2 + \bar{q})(2\bar{r}^4 + \bar{r}^2 + 5\bar{r}^2\bar{q} + 6\bar{q} - \bar{q}^2 - 1)}{2\bar{r}^4 + \bar{r}^2 + \bar{q}\bar{r}^2 - \bar{q}^2 + 2\bar{q} - 1}. \end{aligned} \quad (29)$$

We note that specific heat and susceptibility diverge at

$$2\bar{r}^4 + \bar{r}^2 + \bar{q}\bar{r}^2 - \bar{q}^2 + 2\bar{q} - 1 = 0. \quad (30)$$

This represents the line of continuous phase transition. As one approaches this critical line, correlation length diverges. This shows up, as above, in the divergences of some thermodynamic quantities. Near this critical line, the black holes are expected to exhibit some universal features. These are encoded in a set of critical exponents normally called  $\alpha, \beta, \gamma$  and  $\delta$ . Going close to this line with  $\bar{\mu}$  fixed, we define exponents  $\alpha, \beta, \gamma$  as

$$\bar{C} \sim (\bar{T} - \bar{T}_c)^{-\alpha}, \quad \bar{Q} - \bar{Q}_c \sim (\bar{T} - \bar{T}_c)^\beta, \quad \bar{\chi} \sim (\bar{T} - \bar{T}_c)^{-\gamma}. \quad (31)$$

Here  $\bar{T}_c$  is the value of the critical temperature for the chosen  $\bar{\mu}$  (The critical line can be expressed in terms of  $\bar{T}$  and  $\bar{\mu}$  and is given later, see (41)). Similarly, one defines  $\bar{Q}_c$ . The other static exponent  $\delta$  is defined as

$$\bar{Q} - \bar{Q}_c \sim (\bar{\mu} - \bar{\mu}_c)^{\frac{1}{\delta}}. \quad (32)$$

Here one approaches the critical line with a trajectory on which  $\bar{T}$  is constant. For the black holes in consideration, these quantities are easily calculable and are given by

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2\right). \quad (33)$$

First, note that these exponents are same as those computed for black holes with planar horizon [17, 18]. Secondly, they satisfy the scaling relations

$$\alpha + 2\beta + \gamma = 2, \quad \gamma = \beta(\delta - 1). \quad (34)$$

Our main task is now to construct an effective potential that captures all the phases that we have just discussed. We will use the BW approach for this purpose. This approach requires us to identify an order parameter. Noting the fact that, for a first order transition, the change in order parameter is discontinuous and for second order, it changes continuously, we continue to use the horizon radius  $\bar{r}$  of the black hole as the order parameter. Once a suitable order parameter is identified, one constructs the BW potential which depends on the order parameter, the temperature and the chemical potential. This is given by

$$\bar{\mathcal{F}}(\bar{r}, \bar{T}, \bar{\mu}) = \bar{E} - \bar{T}\bar{S} - \bar{\mu}\bar{Q}. \quad (35)$$

In our case, using (24) and (27), we immediately get<sup>6</sup>

$$\bar{\mathcal{F}}(\bar{r}, \bar{T}, \bar{\mu}) = \frac{1}{2}\bar{r}^2 \left[ 3 - 4\pi\bar{T} \frac{\bar{r}\sqrt{1+\bar{r}^2}}{\sqrt{1+\bar{r}^2}-\bar{\mu}^2} + \bar{r}^2 \left( 3 + \frac{\bar{\mu}^2}{1+\bar{r}^2-\bar{\mu}^2} \right) \right]. \quad (36)$$

The saddle point of  $\bar{\mathcal{F}}$ , namely

$$\frac{\partial \bar{\mathcal{F}}}{\partial \bar{r}} = 0 \quad (37)$$

gives the equilibrium temperature. Using (36), from (37) we get

$$\bar{T} = \frac{\sqrt{1+\bar{r}^2}(1+2\bar{r}^2-\bar{\mu}^2)}{2\pi\bar{r}\sqrt{1+\bar{r}^2-\bar{\mu}^2}}. \quad (38)$$

Upon using (25), the above expression reduces to the one in (26). Furthermore, substituting (38) in (36), we get the on-shell free energy expression as in (28). We now proceed to study  $\bar{\mathcal{F}}$  as we change  $\bar{T}$  and  $\bar{\mu}$ . From the expression of temperature, it is easy to note that it has a minimum  $\bar{T}_0 = 1/\pi$  when  $\bar{r} = 0$  and  $\bar{\mu} = 1$ . In what follows, we will focus ourselves in the region  $\bar{T} \geq \bar{T}_0$  and  $\bar{\mu} \geq 0$ . As noted before, the first order transition line is given by the equation

$$\bar{r}^4 + \bar{\mu}^2 - 1 = 0. \quad (39)$$

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<sup>6</sup>For a black hole with flat horizon a similar construction was provided in [14].

Expressed in terms of  $\bar{T}$  and  $\bar{\mu}$ , this equation reduces to

$$\bar{T} = \frac{2 + \sqrt{1 - \bar{\mu}^2}}{2\pi}, \quad (40)$$

represented by the dotted line in figure 1. On the other hand, the second order instability line (30) reads as

$$\bar{T} = \frac{(-\Delta\bar{\mu}^2 + \Gamma + \Delta)}{\sqrt{2\pi}\Gamma} \sqrt{\frac{\Gamma(\Gamma + 2\Delta)}{\Delta(\Gamma - 2\Delta(\bar{\mu}^2 - 1))}} \quad (41)$$

where

$$\Delta = (\bar{\mu}^2 - 1)^{\frac{2}{3}}(\bar{\mu} + 1)^{\frac{2}{3}}, \quad \Gamma = \bar{\mu}^4 + (\Delta - 2)\bar{\mu}^2 + \Delta^2 - \Delta + 1. \quad (42)$$

This is denoted by the solid line in figure 1.

To see that  $\bar{\mathcal{F}}(\bar{r}, \bar{T}, \bar{\mu})$  captures the whole phase diagram, we first fix  $\bar{\mu}$  and plot  $\bar{\mathcal{F}}$  for various temperatures starting with  $\bar{T} = \bar{T}_0 = 1/\pi$ . We start with  $\bar{\mu} = 1$ . The behaviour is shown in figure 3. We note that at  $\bar{T} = \bar{T}_0 = 1/\pi$ ,  $\bar{\mathcal{F}}$  has a minimum at  $\bar{r} = 0$ . Its first and second derivatives with respect to  $\bar{r}$  also vanish at that point. In this sense, it is a point of inflection for  $\bar{\mathcal{F}}$ . If we increase  $\bar{T}$  further, we get minima for increasing values of  $\bar{r}$  – representing stable black hole phases with increasing size. This is in complete agreement with the phase diagram in figure 1. Next, we analyze the system for  $0 \leq \bar{\mu} < 1$ . From figure 1, we expect that  $\bar{\mathcal{F}}$  should show a HP transition as we increase the temperature beyond a critical value. We precisely see this in figure 4, where we have plotted  $\bar{\mathcal{F}}$  for  $\bar{\mu} = .5$ . While the point  $\bar{r} = 0$  is identified with the AdS phase, any finite value of  $\bar{r}$  represents a black hole with  $\bar{r}$  being the horizon. As we increase the temperature, we note a crossover from AdS to the black hole phase at  $\bar{T} = \bar{T}_{HP} = 1.433/\pi$ . This is shown by the dotted line in the figure. At this temperature the order parameter  $\bar{r}$  changes discontinuously from zero to a finite value - clearly a signature of a first-order transition. Now as we decrease  $\bar{\mu}$ , HP transition temperature increases. In particular, for  $\bar{\mu} = 0$ ,  $\bar{T}_{HP} = 3/(2\pi)$  as expected for AdS-Schwarzschild black hole. Finally, we increase  $\bar{\mu}$  beyond 1. For  $\bar{\mu} = 2$ ,  $\bar{\mathcal{F}}$  is shown in figure 5. Plots are shown for different temperatures, starting with the critical one (solid curve). Below this temperature, we reach the yet unknown phase and the black hole is unstable. At higher temperatures (dashed and dotted curve), minima of the curves represent the stable black hole phases.

We can continue the same exercise for  $\bar{T}$  fixed at any value above  $1/\pi$  and change  $\bar{\mu}$ . For  $\bar{T}_{HP} \leq 3/(2\pi)$  and  $\bar{\mu} \leq 1$ , we first cross the HP line. Close to this point,  $\bar{\mathcal{F}}$  behaves similar to that of figure 4. Further increasing  $\bar{\mu}$  but keeping  $\bar{T}$  fixed, we hit the continuous phase transition line leading to figure 5. For  $\bar{T} \geq 3/(2\pi)$ , the first order transition is lost. Black hole is always a stable phase for low  $\bar{\mu}$ . However, as we take  $\bar{\mu}$  to a critical value, black hole ceases to be stable and we reach the second order line getting a figure similar to figure 5.

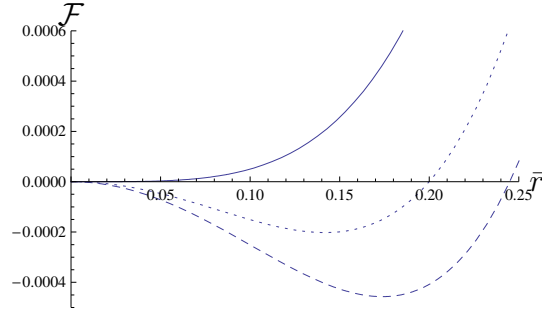


Figure 3:  $\bar{\mathcal{F}}$  is plotted as a function of the order parameter  $\bar{r}$  for  $\bar{\mu} = 1$ . The solid, dotted and dashed curves are for  $\bar{T} = 1/\pi, 1.01/\pi, 1.015/\pi$ .

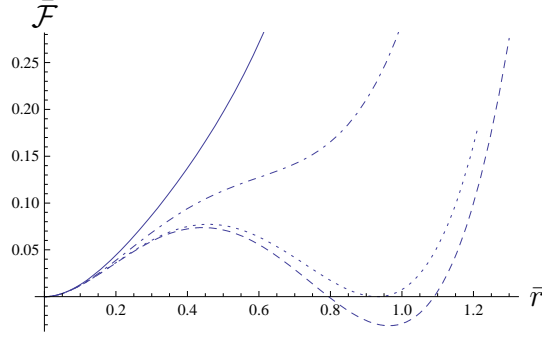


Figure 4:  $\bar{\mathcal{F}}$  is plotted as a function of the order parameter  $\bar{r}$  for  $\bar{\mu} = .5$ . The solid, dot-dashed, dotted and dashed lines are for  $\bar{T} = 1/\pi, 1.3/\pi, 1.433/\pi, 1.45/\pi$ .

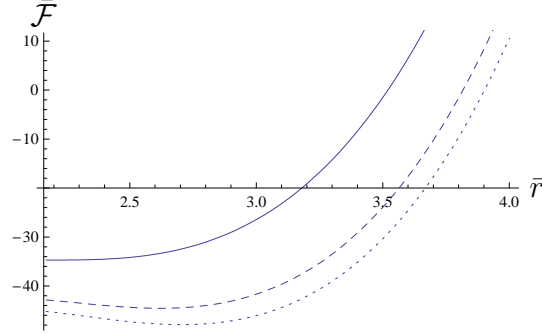


Figure 5:  $\bar{\mathcal{F}}$  for  $\bar{\mu} = 2$ . Solid, dashed and dotted lines are for  $\bar{T} = 0.86, 0.93, 0.95$  respectively. Solid line represents  $\bar{\mathcal{F}}$  at critical temperature. Below this temperature, black hole becomes unstable. The minima in the rest two curves show the stable black hole phase.

Finally, let us now discuss about the procedure for obtaining the critical exponents from the mean field potential  $\bar{\mathcal{F}}$  which has already been written in (33). We note that the specific heat at fixed chemical potential can be obtained from (36).

$$\bar{C}_{\bar{\mu}} = -\bar{T} \left. \frac{\partial^2 \bar{\mathcal{F}}}{\partial \bar{T}^2} \right|_{\bar{\mu}} \sim (\bar{T} - \bar{T}_c)^{-\frac{1}{2}}, \quad (43)$$

which gives  $\alpha = \frac{1}{2}$ . If we approach the critical line along constant  $\bar{\mu} = \bar{\mu}_c$ , then we see that

$$\bar{Q} - \bar{Q}_c \sim (\bar{T} - \bar{T}_c)^{\frac{1}{2}}, \quad (44)$$

where  $\bar{Q}_c$  is the critical value of the charge at fixed  $\bar{\mu}_c$ . This shows that the critical exponent  $\beta$  has the value  $\frac{1}{2}$ . Similarly, the susceptibility behaves near the critical temperature as

$$\chi = \left. \frac{\partial \bar{Q}}{\partial \bar{\mu}} \right|_{\bar{T}} \sim (\bar{T} - \bar{T}_c)^{-\frac{1}{2}}. \quad (45)$$

This leads us to the critical exponent  $\gamma = \frac{1}{2}$ . Finally, on approaching the critical line with  $\bar{T} = \bar{T}_c$  we get

$$\bar{Q} - \bar{Q}_c \sim (\bar{\mu} - \bar{\mu}_c)^{\frac{1}{2}}. \quad (46)$$

So, this gives us  $\delta = 2$ .

## 4 Hairy to Reissner-Nodström black holes: a continuous phase transition

In this section, we first review the main features of the hairy black holes [11] and their instability. We then characterize this instability via BW construction and argue that this black holes undergo a continuous transition at high temperature.

We consider four dimensional gravity action in the presence of a negative cosmological constant where the matter content is given by a conformally coupled real self interacting scalar field and a Maxwell gauge field.

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{16\pi} \left( R + \frac{3}{l^2} \right) - \frac{F_{\mu\nu} F^{\mu\nu}}{16\pi} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} R \phi^2 - \alpha \phi^4 \right). \quad (47)$$

The black holes of this theory are described by the metric

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\sigma^2, \quad (48)$$

with

$$V(r) = \frac{r^2}{l^2} - \left( 1 + \frac{M}{r} \right)^2. \quad (49)$$

In the expression of the metric,  $d\sigma^2$  represents the line element of a constant negative curvature two dimensional manifold. The scalar and the non-zero component of the electromagnetic field are given by

$$\phi = \sqrt{\frac{1}{2\alpha l^2}} \left( \frac{M}{r+M} \right), \quad A_t(r) = -\frac{q}{r}. \quad (50)$$

It is important to note that the mass and charge are not independent but related via

$$q^2 = M^2 \left( \frac{2\pi}{3l^2\alpha} - 1 \right) = M^2(a - 1). \quad (51)$$

Here  $a$  is defined as

$$a = \frac{2\pi}{3l^2\alpha}. \quad (52)$$

In terms of appropriately scaled variables, the temperature, chemical potential, internal energy, charge, and entropy densities are given by [11]

$$\begin{aligned} \bar{T} &= \frac{1}{2\pi}(2\bar{r} - 1), \quad \bar{\mu} = \frac{\bar{q}}{\bar{r}}, \\ \bar{E} &= \frac{1}{4\pi}\bar{r}(\bar{r} - 1), \quad \bar{Q} = \frac{\bar{q}}{4\pi}, \\ \bar{S} &= \frac{\bar{r}^2}{4} \left( 1 - \frac{a(\bar{r} - 1)^2}{\bar{r}^2} \right). \end{aligned} \quad (53)$$

Note that due to the conformal coupling of the scalar to the curvature, the entropy density gets modified from standard form by an “effective” gravitational constant [12]. We also note that entropy remains positive only in the temperature range

$$\frac{1}{2\pi} \left( \frac{\sqrt{a} - 1}{\sqrt{a} + 1} \right) \leq \bar{T} \leq \frac{1}{2\pi} \left( \frac{\sqrt{a} + 1}{\sqrt{a} - 1} \right). \quad (54)$$

We call the limiting values to be  $\bar{T}_{\min}, \bar{T}_{\max}$  respectively.

There is an additional black hole solution to the action (47). We will call this the Reissner-Nordström solution. The metric has the form [11]

$$ds^2 = -V(\rho)dt^2 + V(\rho)^{-1}d\rho^2 + \rho^2 d\sigma^2, \quad (55)$$

with

$$V(\rho) = \frac{\rho^2}{l^2} - \left( 1 + \frac{2M_0}{\rho} - \frac{q_0^2}{\rho^2} \right). \quad (56)$$

with

$$\phi = 0, \quad \text{and} \quad A_t = -\frac{q_0}{\rho}. \quad (57)$$

The event horizon is located at  $V(\rho) = 0$ , the solution of which we will call  $\rho_+$ . Thermodynamic quantities associated with this black holes are

$$\begin{aligned}\bar{T} &= \frac{1}{2\pi} \left( \frac{3}{2} \bar{\rho} - \frac{1}{2\bar{\rho}} - \frac{\bar{q}_0^2}{2\bar{\rho}^3} \right), \\ \bar{E} &= \frac{1}{8\pi} \left( \bar{\rho}^3 - \bar{\rho} - \frac{q_0^2}{\bar{\rho}} \right), \\ \bar{Q} &= \frac{\bar{q}_0}{4\pi}, \quad \bar{S} = \frac{\bar{\rho}^2}{4}, \quad \bar{\mu} = \frac{\bar{q}_0}{\bar{\rho}}.\end{aligned}\tag{58}$$

In the following, we will argue that the hairy black holes, in the grand canonical ensemble, are unstable and crosses over to the RN black holes at high temperature. We will also characterize this instability via BW analysis. First of all, in order to compare two different black holes, namely the RN and the hairy one, we will have to make sure that they have the same temperature and chemical potential. That means

$$\begin{aligned}\frac{1}{2\pi} \left( \frac{3}{2} \bar{\rho} - \frac{1}{2\bar{\rho}} - \frac{\bar{q}_0^2}{2\bar{\rho}^3} \right) &= \frac{1}{2\pi} (2\bar{r} - 1), \\ \frac{\bar{q}_0}{\bar{\rho}} &= \frac{\bar{q}}{\bar{r}}.\end{aligned}\tag{59}$$

These two equations allow us to express  $\bar{q}_0$  and  $\bar{\rho}$  in terms of  $\bar{q}$  and  $\bar{r}$ . In particular, for  $\bar{\rho}$ , we get

$$\bar{\rho} = \frac{1}{3\bar{r}} \left( -\bar{r} + 2\bar{r}^2 + \sqrt{3\bar{q}^2 + 4\bar{r}^2 - 4\bar{r}^3 + 4\bar{r}^4} \right).\tag{60}$$

The BW free energy density for both the black holes can now be easily computed as was done in the previous sections. For the hairy one it reads

$$\begin{aligned}4\pi\bar{\mathcal{F}}_{\text{hair}} &= 4\pi(\bar{E} - \bar{T}\bar{S} - \bar{Q}\bar{\mu}) \\ &= \bar{r}(\bar{r} - 1) - \pi\bar{r}^2 \left( 1 - a \frac{(\bar{r} - 1)^2}{\bar{r}^2} \right) \bar{T} - \bar{r}(\bar{r} - 1) \sqrt{a - 1} \bar{\mu} \\ &= \bar{r}(\bar{r} - 1) - \pi\bar{r}^2 \left( 1 - a \frac{(\bar{r} - 1)^2}{\bar{r}^2} \right) \bar{T} - \frac{(a - 1)}{2} \bar{r}(\bar{r} - 1) (2\pi\bar{T} - 1).\end{aligned}\tag{61}$$

In going from the first line to the second, we use the fact that for hairy black holes,  $\bar{q}$  is not independent, but related to  $\bar{\mu}$  and hence  $\bar{r}$  through (51). Similarly, the conjugates  $\bar{\mu}$  is related to  $\bar{T}$  via

$$\bar{\mu} = \frac{1}{2} \sqrt{a - 1} (2\pi\bar{T} - 1).\tag{62}$$

We used this equation to get to the last line of (61). As for RN black holes, we can proceed similarly to get

$$4\pi\bar{\mathcal{F}}_{\text{RN}} = 4\pi(\bar{E} - \bar{T}\bar{S} - \bar{Q}\bar{\mu})$$



$$\begin{aligned}
&= \frac{\bar{\rho}^3}{2} - \frac{\bar{\rho}}{2} + \frac{q_0^2}{2\bar{\rho}} - \pi\bar{\rho}^2\bar{T} - \bar{q}_0\bar{\mu} \\
&= \frac{\bar{\rho}^3}{2} - \frac{\bar{\rho}}{2} + \frac{\bar{\rho}}{2}(a-1)(\bar{r}-1)^2 - \pi\bar{\rho}^2\bar{T} - \frac{\bar{\rho}}{2}(a-1)(\bar{r}-1)(2\pi\bar{T}-1), \quad (63)
\end{aligned}$$

where we need to substitute  $\bar{\rho}$  using (60) and further  $\bar{q}$  by  $\bar{m}$  and hence by  $\bar{r}$ . In order to write (63), we have also made use of the second identification given in (59). Further, using (60) and (61), after some simplification, we can re-write  $\bar{\mathcal{F}}_{\text{RN}}$  as

$$\begin{aligned}
\bar{\mathcal{F}}_{\text{RN}}(\bar{r}, \bar{T}, \bar{a}) &= \frac{1}{54} \left[ (1+\bar{r})(-1-\bar{r}+\delta)(-5+4\bar{r}-6\pi\bar{T}) + 3a(\bar{r}-1)\{3+12\bar{r}^2-\delta \right. \\
&\quad \left. + 30\pi\bar{T}-6\pi\delta\bar{T}+\bar{r}(-21+4\delta-18\pi\bar{T})\} \right] + \bar{r}(\bar{r}-1) \\
&\quad - \pi\bar{r}^2 \left( 1 - a \frac{(\bar{r}-1)^2}{\bar{r}^2} \right) \bar{T} - \frac{(a-1)}{2} \bar{r}(\bar{r}-1)(2\pi\bar{T}-1). \quad (64)
\end{aligned}$$

with

$$\delta = \sqrt{3a(\bar{r}-1)^2 + (\bar{r}+1)^2}. \quad (65)$$

The saddle point of (61) and (64) occurs at

$$\bar{r} = \frac{1}{2}(1+2\pi\bar{T}), \quad (66)$$

and at the minima,

$$\begin{aligned}
\bar{\mathcal{F}}_{\text{hair}} &= -\frac{1}{8\pi}(\bar{r}^2 + a(\bar{r}-1)^2), \\
\bar{\mathcal{F}}_{\text{RN}} &= \frac{1}{216\pi} \left( 2 + 6\bar{r} - 21\bar{r}^2 + 2\bar{r}^3 - 2\delta(1+\bar{r})^2 - 3a(-1+\bar{r})^2(-3+2\delta+6\bar{r}) \right). \quad (67)
\end{aligned}$$

While for  $\bar{T} \leq \bar{T}_c = 1/(2\pi)$ ,  $\bar{\mathcal{F}}_{\text{hair}}$  minimizes the free energy, for  $\bar{T} \geq \bar{T}_c$ , RN represents the stable black holes. From (66), it follows that at the critical temperature  $\bar{T}_c$ ,  $\bar{r} = \bar{r}_c = 1$ . Near  $\bar{T}_c$  it follows that

$$\bar{r} - \bar{r}_c \sim \bar{T} - \bar{T}_c, \quad \bar{\mathcal{F}} - \bar{\mathcal{F}}_c \sim (\bar{T} - \bar{T}_c)^3, \quad (68)$$

where  $\bar{\mathcal{F}}_c$  is the value of  $\bar{\mathcal{F}}$  at  $\bar{r} = \bar{r}_c$ . The derivative of specific heat with respect to temperature has a discontinuity around  $\bar{r}_c$  of  $(2+a)\pi^2$ . This is thus a continuous phase transition from hairy to RN black holes. The critical exponent following from (68) is  $\alpha = -1, \beta = 1$ . In figure 6, we have plotted  $\bar{\mathcal{F}}$  for different black holes at different temperatures and scalar couplings. The behaviour of  $\bar{\mathcal{F}}$  close  $\bar{T}_c$  is shown in figure 7.

We note that the BW free energy constructed in (61) can also be expressed using the value of the scalar  $\phi$  at the horizon as order parameter. Inverting (50), we can

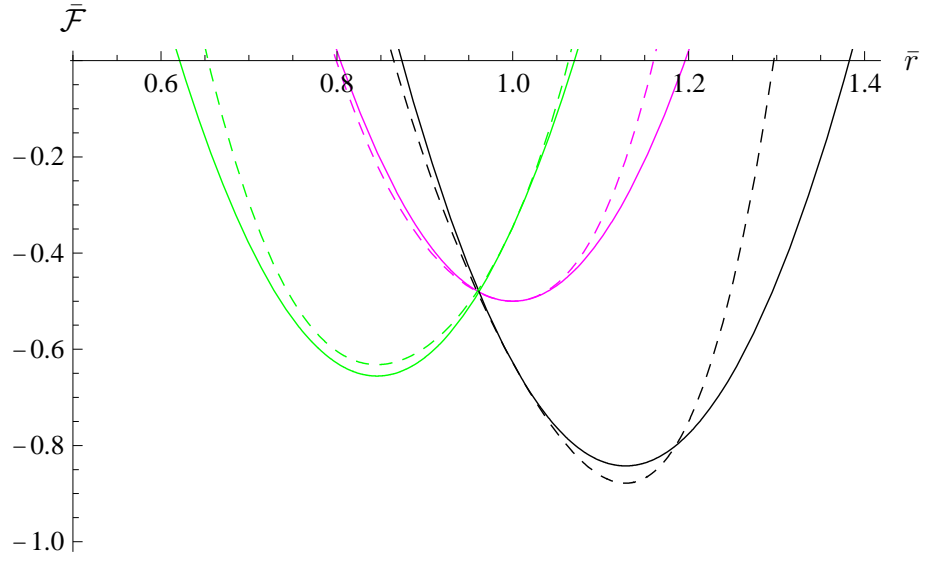


Figure 6: This figure is a plot of (61) and (64) for  $a = 25$  and for different temperatures. The solid lines and the dashed lines represent the hairy and the RN black holes respectively. Green, magenta and black curves are for  $\bar{T} = 0.11, 1/(2\pi), 0.2$  respectively. We see that while at low temperature free energy is minimized by the hairy black hole, RN black holes dominate at high temperature. At  $\bar{T} = 1/(2\pi)$ , free energies are equal at the minimum.

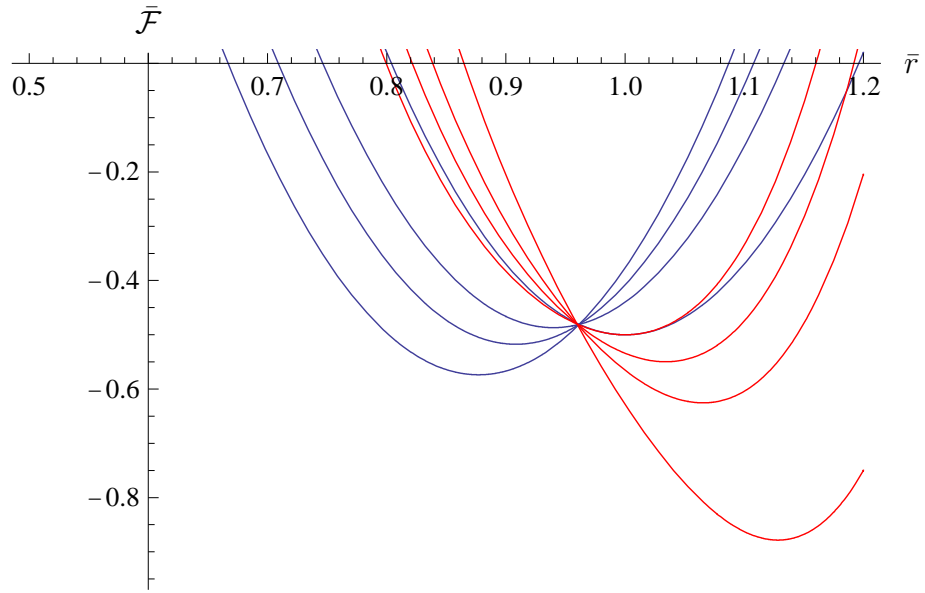


Figure 7: This figure is the behaviour of the free energy function close to  $\bar{T} = \bar{T}_c = 1/(2\pi)$  for  $a = 30$ . Blue and red are for  $\bar{T} \leq T_c$  and  $\bar{T} \geq \bar{T}_c$  representing hairy and RN black holes respectively. At  $\bar{T} = \bar{T}_c$ , the minima for both are degenerate. Clearly, the order parameter  $\bar{r}$ , at which the minima occur, changes continuously around critical temperature leading to a continuous phase transition.

express  $\bar{\mathcal{F}}_{\text{hair}}$  as,

$$4\pi\bar{\mathcal{F}}_{\text{hair}} = \frac{\sqrt{a}}{(\sqrt{\frac{3a}{\pi}} - 2\phi_h)^2} \left( 4\sqrt{a}\pi\bar{T}\phi_h^2 + \sqrt{\frac{3}{\pi}}(1 + a - 2(a-1)\pi\bar{T})\phi_h - 3\sqrt{a}\bar{T} \right). \quad (69)$$

Here  $\phi_h$  is the value of the scalar at the horizon. The expression on the right has a minimum at

$$\phi_h = \sqrt{\frac{3a}{4\pi}} \left( \frac{2\pi\bar{T} - 1}{2\pi\bar{T} + 1} \right), \quad (70)$$

such that, for  $\bar{T} = \bar{T}_c$ ,  $\phi = 0$ .

Having reached this far, we now like to address some dynamical issues associated with this system. In particular, we ask as to how the order parameter  $\phi_h$  behaves in time as we temperature quench the system from  $\bar{T} > \bar{T}_c$  to  $\bar{T} < \bar{T}_c$ . We assume that, during the quench, the temperature changes so fast that  $\phi_h$ , immediately after the change, is identical to its value before. However, at a later time  $\phi_h$  must roll down to its stable position given by (70). In the following, we will be interested in finding out the interpolating solution  $\phi_h(t)$  which connects the unstable to the stable point.

The equation that we need to solve is

$$\partial_t^2 \phi_h(t) + \frac{\partial \bar{\mathcal{F}}_{\text{hair}}}{\partial \phi_h(t)} = 0, \quad (71)$$

where  $\bar{\mathcal{F}}_{\text{hair}}$  is given by (69). This equation can be immediately integrated once to get

$$\frac{1}{2}(\partial_t \phi_h)^2 + \bar{\mathcal{F}}_{\text{hair}}(\phi_h) = C. \quad (72)$$

The integration constant  $C$  can be fixed by the boundary condition  $\partial_t \phi_h = 0$  for  $\phi_h = 0$ . This gives

$$\frac{1}{2}(\partial_t \phi_h)^2 + \bar{\mathcal{F}}_{\text{hair}}(\phi_h) = \bar{\mathcal{F}}_{\text{hair}}(0) \quad (73)$$

It turns out that this equation can be integrated exactly with the boundary condition  $\phi_h(t) = 0$  at  $t = 0$ . The result can be expressed in a form

$$f(\phi_h, a, \bar{T}) = t, \quad (74)$$

where  $f$  is a known function of  $\phi_h$ . Furthermore, it has parametric dependences on  $\bar{T}$  and  $a$ . This function is too non-illuminating and hence we do not display it here. It however turns out that the equation above can not be analytically inverted to get  $\phi_h(t)$  as an explicit function of  $t$ . Nevertheless, numerically it can be solved and the result is shown in the figure 8. In the plot, we have shown two cases where temperature  $\bar{T}$  is quenched down to .14 (red) and .13 (blue). The value of  $a$  that we have chosen is 90. Starting from  $\phi_h(t) = 0$  at  $t = 0$ ,  $\phi_h(t)$  rolls down to respective stable points dictated by the equation (70).

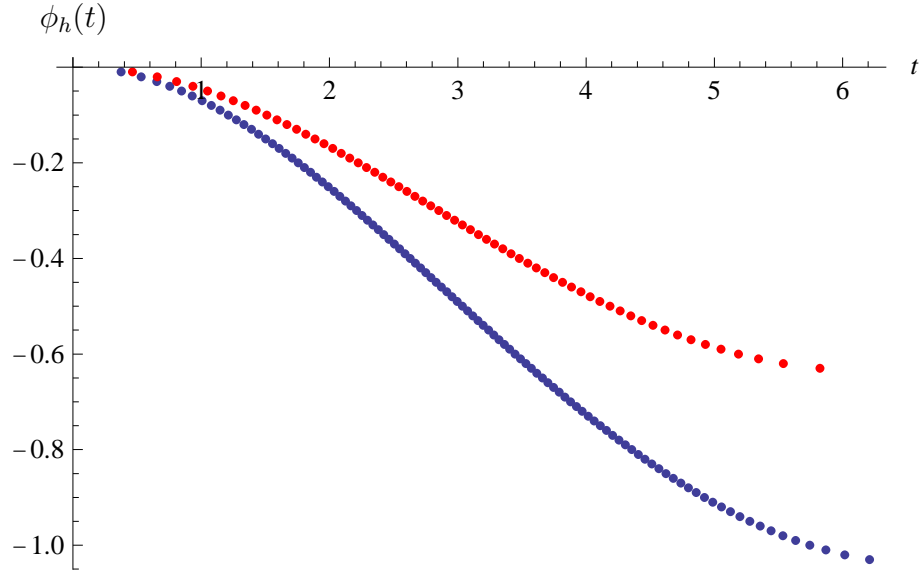


Figure 8: This figure shows behaviour of  $\phi_h(t)$  after quenched to different temperatures below  $\bar{T}_c = 1/(2\pi)$ . The vertical axis is  $\phi_h$  and the horizontal one is  $t$ . The plots are for  $a = 90$ . While the lower one (blue) curve is for temperature quenched to  $\bar{T} = .13$ , the upper one (red) is for  $\bar{T} = .14$ . We see  $\phi_h(t)$  starts with zero value at  $t = 0$  and at a later time reaches a non-zero negative stable point determined by the equation (70)

## 5 Summary

In this work our aim was to study black hole instabilities within the framework of BW theory of phase transition. After providing a pedagogical review to this subject, we employed BW method in two cases. One involved the  $R$ -charged black holes with spherical horizon in five dimensional AdS space. In the presence of non-zero chemical potential, it undergoes both first and second order transitions. We found that BW theory, with horizon radius as order parameter, captures all these instabilities. We hope that, via AdS/CFT correspondence, the constructed BW free energy will be useful to study the phases of strongly coupled  $\mathcal{N} = 4$  SYM theory on  $R^3$  at finite temperature and chemical potential in the same way as in [14].

The other example that we studied is the fate of four dimensional hairy black holes with hyperbolic horizon. Again, via a BW analysis we argued that with the increase in temperature, this black hole becomes unstable, loses its “hair” and turns into a stable RN black hole. This transition is analogous to a third order phase transition with a singularity in the derivative of the specific heat. The BW free energy is constructed in (61). Using value of the scalar on the horizon as order parameter, we studied its behaviour under temperature quench. The corresponding rolling down solutions were semi-analytically constructed.

Within the AdS/CFT correspondence, in [19, 20], second order instabilities associated with hairy black holes with flat horizon were used to understand holographic superconductors at the boundary. We note that superconductors with possible higher order transition (similar to the one we discussed) has been reported earlier, see for example [21]. We hope a construction like (69) will be useful to analyse such holographic superconductors, however in hyperbolic space.

## Acknowledgment

We have benefited from discussions with Swarnendu Sarkar and Goutam Tripathy.

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